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Free-Surface Stress Conditions for Incompressible-Flow Calculations[1]

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ABSTRACT

The numerical study of transient incompressible fluid flows is greatly complicated by the presence of free surfaces. One method of treating such problems is the Marker-and-Cell technique, which in its original form, used simple approximations for the free-surface boundary conditions. These approximations are found to be inaccurate at low Reynolds numbers ($R \lesssim 10$). With a simple modification it is possible to approximate the complete normal stress condition. This modification is shown to have a pronounced effect on some low-Reynolds-number flows.

I. INTRODUCTION

The numerical solution of viscous fluid flow problems is complicated by the presence of free surfaces. There are two reasons for this. First, there must be some means of recording the position of the free surface, and second, the free-surface boundary conditions must be imposed. We shall discuss the free surface problem in connection with the Marker-and-Cell (MAC) computing method [1].

The MAC method is a finite difference technique for solving the time-dependent Navier–Stokes equations. These equations for two-dimensional flows are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} = -\frac{\partial \varphi}{\partial x} + g_x + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{1}$$

$$\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} = -\frac{\partial \varphi}{\partial y} + g_y + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right),$$

where φ is the ratio of pressure to constant density, g_x and g_y are the x and y components of body acceleration, and ν is the kinematic viscosity coefficient. The

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MAC method is based on an Eulerian network of rectangular cells, with flow variables recorded at the locations shown in Fig. 1. Just as the differential equations of motion (1) are statements of the conservation of mass and momentum, the MAC finite-difference equations express these conservation principles for each cell, or combination of cells, in the computing mesh. Complete details of the difference equations can be found in Reference [1]. They will not be needed here.

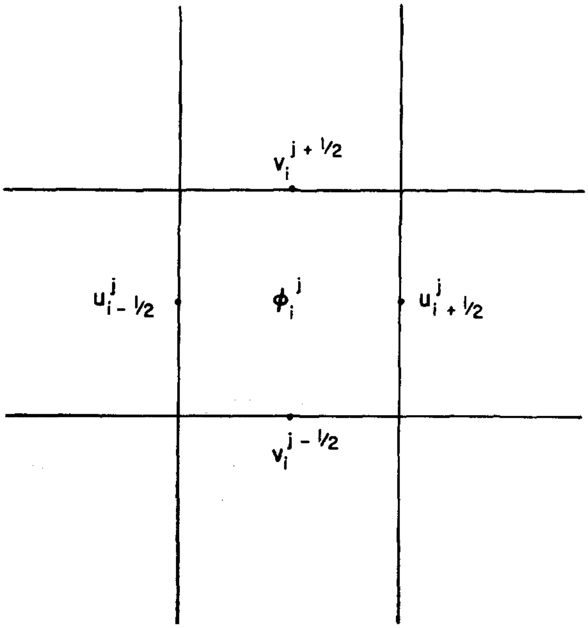

FIG. 1. Typical cell arrangement for MAC.

In addition to the flow variables recorded for each Eulerian cell, there are also recorded the coordinates of selected points distributed throughout the fluid. These coordinates can be thought of as belonging to massless particles that move about with the fluid. They are referred to as *marker* particles, since they mark the flow of fluid much as aluminum dust or hydrogen bubbles are used in an actual laboratory experiment. Thus, the marker particles used in MAC solve the first of the free-surface problems; they indicate which Eulerian cells contain the surface.

The second problem, that of satisfying the free-surface boundary conditions, is much harder. The boundary conditions are that the normal and tangential stresses at the surface must vanish [2]. To satisfy these conditions correctly, the slope of the surface must be known. The MAC technique in its original form used simple approximations for these conditions. The normal stress condition was replaced by a zero surface pressure. This is correct only in the limit of zero viscosity. The tangential stress condition was replaced by two conditions: the fluid incompressibility in surface cells, and the vanishing of the normal derivative of the fluid velocity tangential to the surface.

In this paper we see that these approximate boundary conditions are adequate except for very-low-Reynolds-number flows. A simple modification is described for improving the normal stress condition for small Reynolds numbers, and two examples are shown illustrating the effects of this improvement.

If it is desired to include surface tension forces, then one must also know the curvature of the surface. This difficult problem is not treated here, but is considered in Reference [3].

In Section II a brief description is given of the flow problem used to study the free-surface boundary conditions. Computer-generated results show the failure of the approximate boundary conditions at low Reynolds numbers. In Section III the improved free-surface treatment is described and applied to two examples.

II. The Viscous Bore

A problem that nicely illustrates the effects of free-surface boundary conditions is the formation of a viscous bore. As an idealization, a bore is a step discontinuity in the surface height of an incompressible fluid. A bore is the incompressible analog of a shock wave in a compressible fluid. The jump conditions relating uniform fluid states on either side of the bore are derived from the requirements of mass and momentum conservation. Referring to Fig. 2 these conditions are [4]

$$\begin{aligned} u_1 &= \left(1 - \frac{h_0}{h_1}\right) V, \\ V &= \left[g \frac{h_0}{h_1} \left(\frac{h_0 + h_1}{2}\right)\right]^{1/2}, \end{aligned} \tag{2}$$

where g is the downward acceleration of gravity.

As a consequence of (2) it is easy to show that the fluid must lose kinetic energy in the bore transition. Usually, the energy lost is accounted for by the presence of turbulence at the bore front. The tendency to a more random velocity field behind the bore is analogous to the increase in entropy in a shock transition. The turbulence kinetic energy is continually transformed into heat by the action of molecular viscosity. However, at sufficiently low Reynolds numbers, there can be enough viscous dissipation of mean kinetic energy in a bore to preclude the development of turbulence and keep the flow laminar.

To test the MAC method, we studied the formation of bores with Reynolds

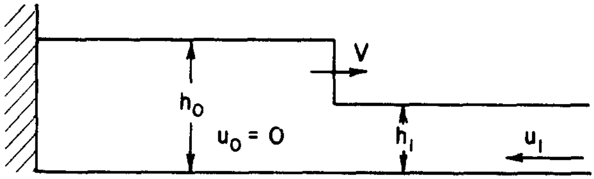

FIG. 2. The idealized bore.

numbers from 4.33 to 79.25. (Reynolds numbers refer to the upstream flow height and upstream velocity of flow relative to the bore front.)

In the calculations reported the fluid was allowed to slip freely along the bottom boundary of the computational mesh. Although this does not correspond to reality at the low Reynolds numbers considered, it does yield a useful test problem for checking the MAC method. It isolates one of the two ways in which viscosity can manifest itself; a procedure difficult to accomplish experimentally.

Figure 3 shows the fluid configuration of a bore at $R = 79.25$. Large eddies are apparent at the bore front, indicating the early stages in transition to turbulence. The calculational results, however, do not depict true turbulence for two reasons.

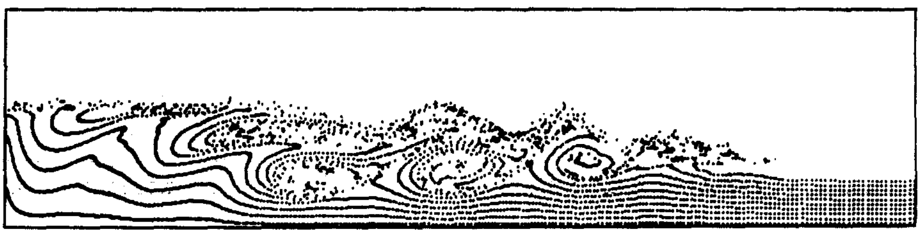

FIG. 3. Bore at $R = 79.25$ showing large eddy structure. Configuration shown at $t = 11.00$.

First, the resolution of the computation region is too coarse, 23 by 100 cells. Second, the calculations are two-dimensional while true turbulence is intimately dependent on three-dimensional motion. Even so, the bore has the correct height and speed expected from the ideal theory (2).

The results for the $R = 79.25$ bore, Fig. 3, are to be compared with those in Fig. 4, which show a bore at $R = 4.33$. The flow in this instance is clearly laminar. The transition from laminar to turbulent flow occurs at a Reynolds number between 20 and 30.

Unfortunately, the results shown in Fig. 4 are not correct. According to (2), the height of the fluid behind the bore should be 1.5, while the value calculated is 1.4. Also the bore speed should be 0.577, which is less than the calculated value, 0.652. These results show that momentum is not conserved. The source of this discrepancy lies in the incorrect treatment of the free-surface boundary conditions.

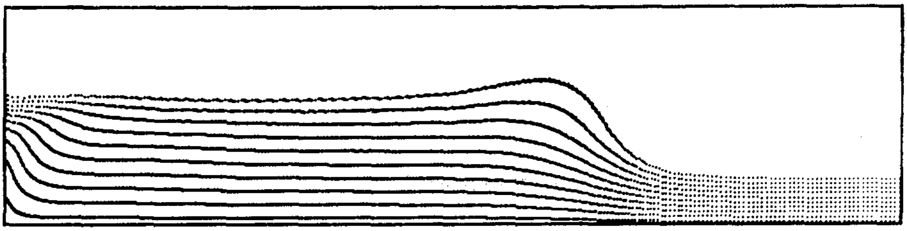

FIG. 4. Bore at $R = 4.33$ calculated without free surface correction. Configuration shown at $t = 11.00$.

III. FREE-SURFACE BOUNDARY CONDITIONS

The correct free-surface boundary conditions are the vanishing of the normal and tangential stresses. To express these conditions in differential form we need the stress tensor for an incompressible fluid [2]

$$\sigma_{ij} = -\varphi\delta_{ij} + \nu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right), \tag{3}$$

where δ_{ij} is the Kronecker delta. Physically $-\sigma_{ij}$ is the amount of ith-component momentum flowing per unit time through unit area normal to the jth direction. Since there is no flux of momentum through a free surface, the boundary condition is

$$\sigma_{ij}n_j = 0, \tag{4}$$

where n_j is a unit normal to the surface.

For a two-dimensional surface, $y = \eta(x, t)$, the boundary conditions (4) take the form

$$\begin{aligned} \varphi - 2\nu\left[n_x n_x \frac{\partial u}{\partial x} + n_x n_y\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + n_y n_y \frac{\partial v}{\partial y}\right] &= 0, \\ \nu\left[2n_x m_x \frac{\partial u}{\partial x} + (n_x m_y + n_y m_x)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + 2n_y m_y \frac{\partial v}{\partial y}\right] &= 0. \end{aligned} \tag{5}$$

The x and y components of the unit outward normal vector to the surface are

$$n_x = \frac{\partial\eta}{\partial x}\left[1 + \left(\frac{\partial\eta}{\partial x}\right)^2\right]^{-1/2},$$

$$n_y = \left[1 + \left(\frac{\partial\eta}{\partial x}\right)^2\right]^{-1/2}.$$

The components of the tangent vector to the surface are

$$m_x = n_y\,, \qquad m_y = -n_x\,.$$

If the curvature of the surface is small we can approximate these conditions by the simpler expressions

$$\varphi - 2\nu(\partial u_n/\partial n) = 0, \tag{6}$$

$$\nu\left(\frac{\partial u_n}{\partial m} + \frac{\partial u_m}{\partial n}\right) = 0, \tag{7}$$

where n refers to the local outward normal direction of the free surface and m to the tangential direction.

In the original MAC method [1] the normal stress condition (6) was simply replaced by $\varphi = 0$. The viscous contribution was omitted. In place of the tangential stress condition (7), the MAC method imposes the condition of incompressibility in each surface cell. This is an approximation, since the latter condition should be imposed only in that region of the surface cell actually occupied by fluid. In addition, when velocities are needed in cells outside the free surface, these velocities are chosen equal to the corresponding velocities inside the surface. The result is an approximation to $\partial u_m/\partial n = 0$, which is not quite the tangential stress condition (7).

Keeping these approximations in mind, and using the physical interpretation of σ_{ij}, we can readily explain why the bore in Fig. 4 is moving too fast. At the front of the bore it is apparent that $\partial u_n/\partial n$ is positive. The calculation used $\varphi = 0$ there, but according to (6) the pressure should be positive. Thus, the effective normal stress at the surface is equal to

$$\sigma_{nn} = 2\nu(\partial u_n/\partial n),$$

which corresponds to a flux of normal momentum into the fluid. This causes the bore to move forward too rapidly. It follows also that the fluid height behind the bore is too low.

It is relatively easy to modify the pressure at the surface to satisfy the correct normal stress condition. There are three cases to consider, which correspond to a surface cell having empty cells on one, two, or three sides.

For a surface cell (i, j) having one side adjacent to an empty cell, say the side at $(i, j + \frac{1}{2})$, see Fig. 1, the velocity on this side is chosen such that $\nabla \cdot \mathbf{u}$ is zero for the cell. [1] The pressure for the cell is then set equal to

$$\varphi_i^j = \frac{2\nu}{\delta y}\,(v_i^{j+1/2} - v_i^{j-1/2}). \tag{8}$$

For a surface cell with three open sides or for a cell with two open sides that are opposite one another the pressure for the cell is set equal to zero.

For a cell with two open sides that are adjacent, the outward normal direction is assumed to be at 45° between the open sides. In this case the normal stress condition (5) reduces to

$$\varphi = \pm \nu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{9}$$

where the sign is chosen equal to the sign of $n_x n_y$. The velocity derivatives in (9) are easily approximated by local finite differences. For example, if the open sides are at $(i, j + \frac{1}{2})$ and $(i + \frac{1}{2}, j)$, then n_x and n_y are positive and the pressure for cell (i, j) is given by

$$\varphi_i^{\,j} = \nu \left[\frac{1}{2\delta y}\left(u^j_{i-1/2} + u^j_{i+1/2} - u^{j-1}_{i-1/2} - u^{j-1}_{i+1/2}\right)\right.$$
$$\left. + \frac{1}{2\delta x}\left(v_i^{j+1/2} + v_i^{j-1/2} - v_{i-1}^{j+1/2} - v_{i-1}^{j-1/2}\right)\right]. \tag{10}$$

Similar expressions are easily written down for other combinations of two adjacent open sides.

These modifications of the normal stress conditions were incorporated into the MAC method and the bore calculation in Fig. 4 was repeated. The new results are given in Fig. 5. The bore speed is 0.556, which is now less than the theoretical value, 0.577. The bore front is also flatter than previously calculated, but the overshoot in surface elevation is still present.

It appears likely that this overshoot, and also the slow bore speed, result from inaccuracies in the tangential stress condition. The argument for this is much the same as that used for the normal stress errors. The calculation more or less approximates the tangential stress condition by $\partial u_m/\partial n = 0$. According to (7) this results in a surface tangential stress,

$$\sigma_{mn} \approx \nu(\partial u_n/\partial m).$$

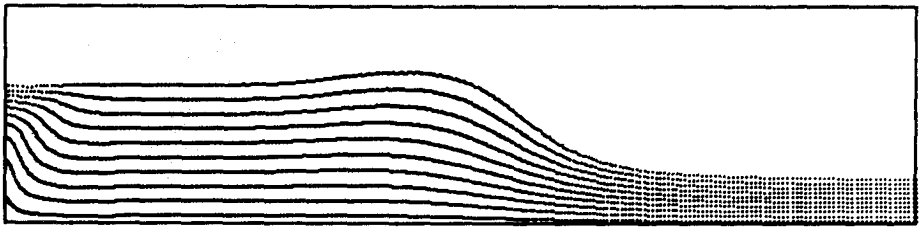

FIG. 5. Bore at $R = 4.33$ calculated with free surface correction. Configuration shown at $t = 11.00$; compare with Fig. 4.

At the bore front $\partial u_n/\partial m$ is negative which means there is a flux of tangential momentum into the surface. Such a flux will slow the bore down and contribute to the overshoot in elevation.

It would, of course, be desirable to devise a way of satisfying both the normal and tangential stress conditions. However, it appears that this requires a more accurate knowledge of surface position and shape than is presently determinable in the basic MAC method. On the other hand, it is emphasized that the free-surface conditions as treated in the original MAC method are accurate for many purposes. Viscous effects on the surface stress are significant only in very-low-Reynolds-number flows; for the bore calculations reported here, the corrections are significant only at Reynolds numbers less than about 10.

An additional illustration of the influence of free-surface boundary conditions is given by the "teapot" effect. Suppose viscous fluid is allowed to run down the side of a wall under the action of gravity. If the wall terminates in a sharp corner,

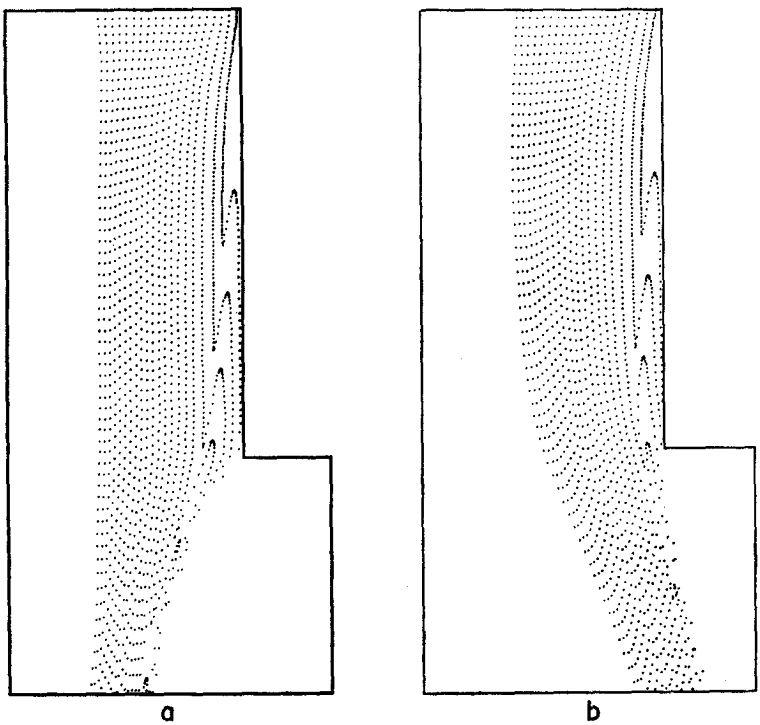

FIG. 6. Teapot effect calculation at $R = 2.0$; (a) without free-surface correction, (b) with free-surface correction.

fluid falling off the wall will bend in underneath the corner, like the dribbling observed at the spout of a teapot.

In Fig. 6a a MAC calculation is shown in which fluid has been input along the top boundary of the computational mesh. The fluid is input with a uniform downward velocity, but it quickly establishes a boundary layer along the wall on its right side. The retarding influence of the wall is just balanced by the downward acceleration of gravity. The fluid fictitiously swings outward from the wall at the corner. However, the calculation neglected the viscous contribution to the normal stress condition. Since the Reynolds number for the flow, based on input mass flux, is 2.0, viscous contributions to the free-surface boundary conditions are not negligible.

Figure 6b shows the same problem repeated with the improved normal stress condition. The flow swings under the corner, exhibiting the "teapot" effect. It is a simple matter to see why this happens. The incorrect result, Fig. 6(a), is caused by a flux of normal momentum through the free surface in the corner region. With the normal stress correction described in this paper, the fluid surface pressure at the corner is negative, and since the surface pressure on the left side of the stream is nearly zero there is a pressure gradient across the stream that pushes it underneath the corner.

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